where the error $\{\epsilon^{(j)}\}$ is simply defined as the difference between the jth iteration and the exact solution. Substitution of Eq. (12) into Eq. (10), then using Eq. (11) in the result, yields

$$\{\delta_B{}^{(j+1)}\} = \{\delta_B\} - [K_P]^{-1}[K_S]\{\epsilon^{(j)}\} \tag{13}$$

These manipulations are repeated; however Eq. (13) is used in place of Eq. (12). By continuing this process, the following equation is obtained

$$\{\delta_B{}^{(j+n)}\} = \{\delta_B\} + (-1)^n \left([K_P]^{-1}[K_S]\right)^n \{\epsilon^{(j)}\} \tag{14}$$

Next, $[K_P]^{-1}$ $[K_S]$ is expressed in terms of its eigenvector matrix $[X]$ with columns $\{x_i\}$ and its eigenvalue matrix $[\Lambda]$ (which is diagonal with the eigenvalues λ_i on the diagonal)

$$[K_P]^{-1}[K_S] = [X]\,[\Lambda]\,[X]^{-1} \tag{15}$$

Since

$$\left([K_P]^{-1}[K_S]\right)^n = [X]\,[\Lambda]^n[X]^{-1} \tag{16}$$

and the eigenvalues of a positive-definite or semidefinite matrix must be non-negative, Eq. (14) shows that the iteration procedure will be convergent for all errors $\{\epsilon\}$ if and only if the magnitudes of all eigenvalues of $[K_P]^{-1}[K_S]$ are less than unity.

To obtain some physical insight into this convergence criterion, assume that deflections corresponding to any eigenvector, $\{x_i\}$, are applied to the secondary structure at the interface points and that these are the only points that are loaded. Equation (3) shows that the load on the secondary structure is $[K_S]\{x_i\}$. Next, suppose that an equal and opposite load is applied to the primary structure. Equation (7) shows that the resulting deflection is

$$[K_P]^{-1}[K_S]\,\{x_i\} = \lambda_i\{x_i\} \tag{17}$$

Thus, the primary-structure deflection is λ_i times the imposed secondary-structure deflection $\{x_i\}$. Consequently, if $0 \le \lambda_i < 1$, the secondary-structure may be considered to be more flexible than the primary structure in the ith mode. Since the convergence criterion requires that $\lambda_{i\max} < 1$, the procedure will be convergent when the secondary structure is "more flexible at the interface boundary", so to speak, than the primary structure.

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On a Linear Time Varying System and Liapunov Criteria for Stability

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I. Introduction

A frictionless second-order system with time varying natural frequency is studied. For such a system analytical approaches exist in the literature dealing with the subject of missiles exiting or entering the atmosphere.[1] The case where the frequency of the system is reduced monotomically with time is the first example, known to the author, in which the solution diverges from its original position, but the system is stable in the sense of Liapunov.

Received June 9, 1975. The work was performed while the author was at University of Notre Dame, Notre Dame, Ind. The author is indebted to R.W. Liu of the University of Notre Dame for his guidance and encouragement.

II. Analytical Approach[1]

Consider the linear time varying system

$$\ddot{x} + N(t)x = 0 \tag{1}$$

where

$$N(t) \ge \epsilon > 0 \text{ for } t \ge 0 \text{ and } x \epsilon R^2$$

Multiply Eq. (1) by $2\dot{x}/N(t)$

$$2\ddot{x}\dot{x}/N(t) + 2x\dot{x} = 0 \tag{2}$$

Integrate between t_1 and t_2 ($t_2 > t_1 >$ to)

$$\int_{t_1}^{t_2} \frac{2\ddot{x}\dot{x}}{N(t)}\,\mathrm{d}t + \int_{t_1}^{t_2} 2x\dot{x}t = 0 \tag{3}$$

The first term in Eq. (3) is integrated by parts

$$\int_{t_1}^{t_2} \frac{\dot{N}(t)}{N^2(t)}\,\dot{x}^2\mathrm{d}t + \frac{\dot{x}^2}{N(t)}\Big|_{t_1}^{t_2} + x^2\Big|_{t_1}^{t_2} = 0 \tag{4}$$

In accordance with the linear theory for constant coefficients assume that the behavior of $x(t)$ is of an oscillatory type, and choose t_1 and t_2 to be at two consecutive peaks, where at a peak:

$$\dot{x} = 0 \tag{5}$$

So, Eq. (4) becomes

$$x^2(t_1) - x^2(t_2) = \int_{t_1}^{t_2} \frac{\dot{N}(t)\dot{x}^2}{N^2(t)}\,\mathrm{d}t \tag{6}$$

From Eq. (6) it is obvious that if $\dot{N}(t) < 0$ then

$$x^2(t_1) < x^2(t_2) \tag{7}$$

Which means that the envelop of $x(t)$ diverges. Similarly for $\dot{N}(t) > 0$ the envelop of $x(t)$ converges.

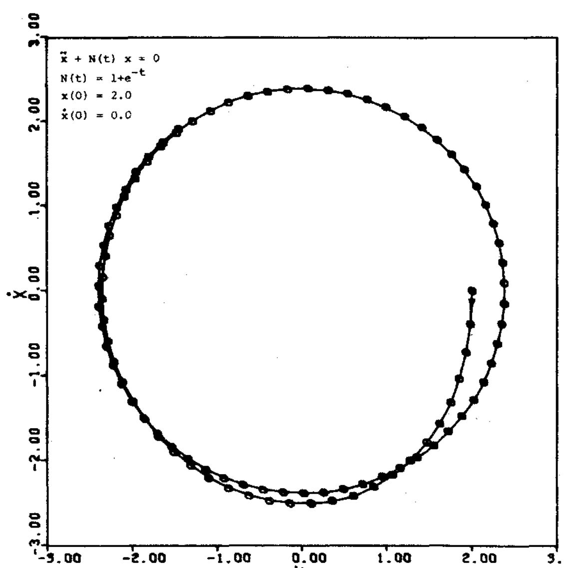

Fig. 1 A trajectory for the decreasing natural frequency case.

III. Liapvnov's Stability Criteria

Define a positive definite Liapvnov function

$$V_l = N(t)x^2 + \dot{x}^2 \tag{8}$$

Establish $\dot{V}_l$ using Eq. (1)

$$\dot{V}_l = \dot{N}(t)x^2 \tag{9}$$

V_l is negative semidefinite for $\dot{N}(t) < 0$, and the system is stable in the sense of Liapvnov. From Eq. (7) it is seen that in this case the motion diverges. As $N(t) > 0$ for all t denote

$$\text{Lim}\quad N(t) = C_l > 0 \tag{10}$$

The motion originated at x_0 diverges to a limit cycle that is approximated by[1]

$$x(\infty) \approx x(0)\sqrt[4]{N(0)/C_l} \tag{11}$$

If C_l *is a very small positive number,* $x(\infty)$, although bounded, becomes very large, and the system is considered to be "unstable" from the practical point of view.

As an example consider the function

$$N(t) = 1 + e^{-t} \tag{12}$$

where the system starts from

$$x(0) = 2.0 \qquad \dot{x}(0) = 0.0 \tag{13}$$

The trajectory is calculated by numerical integration and is presented in Fig. 1.

In a similar way, it can be shown, for completeness, that the system is stable in the sense of Liapvnov for $\dot{N}(t) > 0$. Define

$$V_2 = \dot{x}^2/N(t) + x^2 \tag{14}$$

so,

$$\dot{V}_2 = -(\dot{N}(t)/N^2(t))\dot{x}^2 \tag{15}$$

V_2 is negative semi-sefinite for $\dot{N}(t) > 0$. In this case the motion converges to a limit cycle.

References

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Resonances of a Two-DOF System on a Spin-Stabilized Spacecraft

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Nomenclature

- A = spacecraft transverse moment-of-inertia—ft lb sec^2
- C = spin inertia—ft lb sec^2
- H = angular momentum of spacecraft ($\approx C\omega_z$)—ft lb sec
- R = mz_0^2/A
- c_x, c_y = damping constants in x and y directions—lb sec/ft
- k_x, k_y = spring constants in x and y directions—lb/ft
- m = "reduced mass"† of spring-mass device—lb sec^2/ft
- Ω = nutation frequency of spacecraft—rad/sec
- ξ = normalized x displacement—u/z_0
- η = normalized y displacement—v/z_0
- θ = nutation angle—rad
- $\omega_x, \omega_y, \omega_z$ = angular velocities about coordinate axes—rad/sec

Received June 20, 1975. This work was funded by company-sponsored research while the author was employed by Hughes Aircraft Company.
Index category: Spacecraft Attitude Dynamics and Control.
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Introduction

THIS Note is concerned with the dynamic behavior, particularly natural frequencies and resonances, of a 2 degrees-of-freedom (DOF) spring-mass device on a spin-stabilized spacecraft (Fig. 1). The spring-mass device can move in a plane normal to the spin-axis; this device could be a pendulum, cantilever beam with a tip mass, or an actual spring-mass device.

The equations-of-motion of such a system have been derived many times[2] and are

$$\dot{\omega}_x + \left(\frac{H}{A} - \omega_z\right)\omega_y + R\left[\frac{c_y}{m}\dot{\eta} + \frac{k_y}{m}\eta\right] = 0 \tag{1}$$

$$\dot{\omega}_y - \left(\frac{H}{A} - \omega_z\right)\omega_x - R\left[\frac{c_x}{m}\dot{\xi} + \frac{k_x}{m}\xi\right] = 0 \tag{2}$$

$$\ddot{\xi} + \frac{c_x}{m}(1+R)\dot{\xi} + \left[\frac{k_x}{m}(1+R) - \omega_z^2\right]\xi - 2\omega_z\dot{\eta} + \frac{H}{A}\omega_x = 0 \tag{3}$$

$$\ddot{\eta} + \frac{c_y}{m}(1+R)\dot{\eta} + \left[\frac{k_y}{m}(1+R) - \omega_z^2\right]\eta + 2\omega_z\dot{\xi} + \frac{H}{A}\omega_y = 0 \tag{4}$$

for a symmetrical spacecraft.

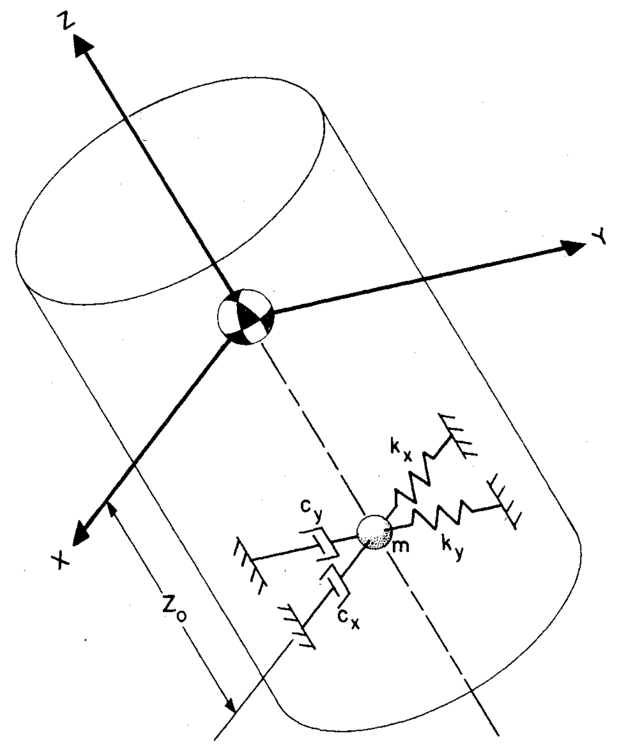

Fig. 1 Two-DOF motion perpendicular to spin axis.

†"Reduced mass" is $m = m_d[1 - (m_d/m_t)]$, where m_d is the actual mass of the device, and m_t is the total spacecraft mass.